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# The relationship between controlled joint torque and end-effector force in underactuated robotic systems

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**SUMMARY**

The *generalized Jacobian matrix* was introduced for dealing with end-effector control in space robots. One of the applications of this Jacobian is to be used in *Jacobian transpose control* to generate joint torques given end-effector position error. It would be misleading, however, to consider the transpose of this Jacobian as a mapping from end-effector force/moment to controlled joint torques for underactuated systems or floating base robots. This paper explains why it does not represent the mapping and provides a simple example. Later, the correct mapping is provided using the dynamically consistent Jacobian inverse and then a method to compute the actuated-joint torques is explained given the desired end-effector force. Finally, the effect of using the generalized Jacobian in the Jacobian transpose control is analyzed.

KEYWORDS: Generalized Jacobian; Space robot; Under-actuation; Torque–force relationship.

## 1. Introduction

The generalized Jacobian matrix[1] was introduced for space robotics and is a reduced dimension Jacobian that relates the end-effector velocities to the joint velocities, excluding base velocities. To remove the columns corresponding to the base coordinate velocities, these velocities are expressed with the joint velocities by using linear and angular momentum conservation. The generalized Jacobian has been used in the Jacobian transpose control to produce joint control torques given end-effector errors.[2-4]

The transpose of this Jacobian, however, does not represent a mapping from end-effector force to joint torque for floating base robots. It can be easily misunderstood as a torque–force relationship for space robots because of the similarity with a proper Jacobian for fixed base manipulators.

This paper explains this problem and gives a simple example to demonstrate it. Later, the correct torque–force relationship is provided and then a method to compute the torques for only actuated joints on the underactuated systems is explained given the desired end-effector force. Finally, the possible limitation of the generalized Jacobian is shown by analyzing the effect of using the generalized Jacobian in Jacobian transpose control.

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## 2. Generalized Jacobian for Jacobian Transpose Control

For the manipulator control of space robots, the generalized Jacobian[1] is used to describe the end-effector velocity in terms of only joint velocities. To derive the generalized Jacobian, we start with Jacobian which includes the base coordinates:

$$\dot{x} = J\dot{q}, \tag{1}$$

where $x$ is the end-effector coordinate, $J$ is Jacobian, and $q$ is the base and joint coordinates. The Jacobian and joint velocities can be divided into the components corresponding to the base and actuated joint coordinates:

$$J = [J_{base} \quad J_{actuated}], \tag{2}$$

$$\dot{q} = \begin{bmatrix} \dot{q}_{base} \\ \dot{q}_{actuated} \end{bmatrix}, \tag{3}$$

where $\dot{q}_{actuated}$ is a vector of joint velocities of the manipulator, excluding the velocity of the base, $\dot{q}_{base}$. Then, the base velocity vector, $\dot{q}_{base}$, can be expressed by $\dot{q}_{actuated}$ using conservation of linear and angular momentum in space robots with zero linear and angular momentum condition:[2–5]

$$\dot{q}_{base} = H\dot{q}_{actuated}, \tag{4}$$

where the matrix, $H$, represents the relationship between the velocities of the actuated joints and those of the base coordinates. From Eqs. (1)–(4), the generalized Jacobian can be defined as

$$\dot{x} = J^{*}\dot{q}_{actuated}, \tag{5}$$

where

$$J^{*} = J_{base}H + J_{actuated}. \tag{6}$$

This generalized Jacobian in Eq. (5) can be used to compute the required joint velocities for desired end-effector velocities. That is,

$$\dot{q}_{actuated} = (J^{*})^{+}\dot{x}. \tag{7}$$

Up to this point, the relationship holds true. Subsequently, the generalized Jacobian was used to simply replace a normal

Jacobian for the Jacobian transpose control:[2–4]

$$\Gamma_{actuated} = (J^*)^{\mathrm{T}} F_{desired}, \tag{8}$$

where $\Gamma_{actuated}$ denotes the vector of the joint torques on the actuated joints and $F_{desired}$ represents the end-effector force that we desire to generate at the end-effector.

## 3. Generalized Jacobian: Incorrect Torque–Force Mapping

The joint torque, $\Gamma_{actuated}$, in Eq. (8) *does not* generate the desired end-effector force, $F_{desired}$. One way to see this is by applying the principle of virtual work. The virtual work can be expressed, by using Eq. (5), as following:

$$\begin{aligned} \delta W &= \delta x^{\mathrm{T}} F - \delta q_a^{\mathrm{T}} \Gamma_a \\ &= \delta q_a^{\mathrm{T}} \{ (J^*)^{\mathrm{T}} F - \Gamma_a \}, \end{aligned} \tag{9}$$

where the subscript $a$ represents *actuated*. Unlike a fixed base robot, however, $\delta W \neq 0$ because the system is not in static equilibrium due to the floating base or underactuation. Indeed, because of the floating base there will be motion resulting in the generation of kinetic energy equivalent to $\delta W$. Therefore, Eq. (8) does not hold as a physical torque–force relationship the way a normal Jacobian does for fixed base manipulators.

## 4. Example of Generalized Jacobian for Torque–Force Mapping

A simple example is presented in this section to demonstrate the incorrect force–torque mapping by the generalized Jacobian. Two point masses are connected by two prismatic joints. The choice of prismatic joints helps to make the math simpler and to easily show the resulting end-effector force. In this example, the term joint torque is actually joint *force* because the joints are prismatic.

In the system in Fig. 1, the first joint force, $\Gamma_1$, will be set to be zero to make the system underactuated. Essentially, the first link acts as a floating base. The second joint and link, in turn, represent the underactuated manipulator. The objective is to compute joint force to produce an end-effector force, $F_{desired}$. Obviously, the answer should be $\Gamma_2 = F_{desired}$ for the second joint's input force.

The Equations of motion of the system are

$$\begin{aligned} m_1 \ddot{q}_1 &= \Gamma_1 - \Gamma_2, \\ m_2(\ddot{q}_1 + \ddot{q}_2) &= \Gamma_2, \end{aligned} \tag{10}$$

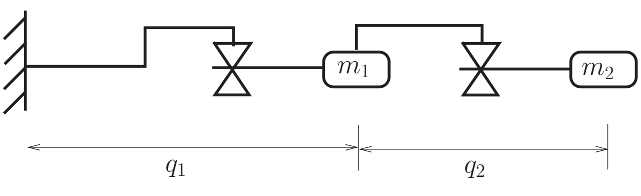


Fig. 1. Two prismatic joints with two mass system.

where $q_1$ is the displacement of $m_1$ (the mass of link 1), and $q_2$ is the displacement from $m_1$ to $m_2$ (the mass of link 2). The terms $\Gamma_1$ and $\Gamma_2$ are the corresponding forces on the masses.

If there is no actuator on the first joint, the system is not fully actuated, i.e. underactuated:

$$\Gamma_1 = 0. \tag{11}$$

This provides a relation between $q_1$ and $q_2$, which is

$$(m_1 + m_2)\ddot{q}_1 + m_2 \ddot{q}_2 = 0. \tag{12}$$

This condition, in fact, can be derived from the conservation of linear momentum. Assuming there are no external forces on the system and zero initial conditions,

$$\dot{q}_1 = -\frac{m_2}{m_1 + m_2} \dot{q}_2, \tag{13}$$

where $-\frac{m_2}{m_1+m_2}$ corresponds to the matrix, $H$, in Eq. (4), i.e. $H = -\frac{m_2}{m_1+m_2}$.

Now, if the second mass, $m_2$, is to be controlled, the corresponding Jacobian is defined as

$$\begin{aligned} \dot{x} &= [1 \quad 1] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\ &= J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}. \end{aligned} \tag{14}$$

Equation (14) can be simplified by using Eq. (13) as:

$$\begin{aligned} \dot{x} &= \frac{m_1}{m_1 + m_2} \dot{q}_2 \\ &= J^* \dot{q}_2. \end{aligned} \tag{15}$$

Using the generalized Jacobian, the velocity of the control point can thus be expressed with only the actuated joint, $\dot{q}_2$.

As before, the relation up to this point holds true. When this is used in the Jacobian transpose control, however, we have

$$\Gamma_2 = (J^*)^{\mathrm{T}} F_{desired}, \tag{16}$$

which results in

$$\Gamma_2 = \frac{m_1}{m_1 + m_2} F_{desired}. \tag{17}$$

The resulting force on the second joint, $\Gamma_2$, is not the desired end-effector force, $F_{desired}$. That is, the generalized Jacobian does not represent the correct torque–force relationship.

## 5. Correct Torque–Force Relationship

For a general robotic system, the control torque, $\Gamma$, to produce a desired end-effector force, $F$, is[6]

$$\Gamma = J^{\mathrm{T}} F + N^{\mathrm{T}} \Gamma_0, \tag{18}$$

where $\Gamma_0$ is an arbitrary torque vector. The matrix, $N^{\mathrm{T}}$, is the null-space projection matrix:

$$\begin{aligned} N^{\mathrm{T}} &= I - J^{\mathrm{T}}\bar{J}^{\mathrm{T}}, \\ \bar{J} &= A^{-1}J^{\mathrm{T}}\Lambda, \\ \Lambda &= (JA^{-1}J^{\mathrm{T}})^{-1}, \end{aligned} \tag{19}$$

where the matrix $A$ represents the joint space inertia matrix. The notation $\overline{(.)}$ represents the dynamically consistent inverse of the quantity. And the inverse relationship from the torque to the end-effector force is[6]

$$F = \bar{J}^{\mathrm{T}}\Gamma. \tag{20}$$

Note that Eq. (20) computes the end-effector force, $F$, given the joint torque, $\Gamma$. This equation holds whether some of the joints are underactuated or not since the particular components would simply be zero. However, Eq. (18) computes the required torques to generate the desired end-effector force, $F$. The computed torque cannot be applied to the robot *if* there is any unactuated joint: for example, the floating base for space robots.

For the underactuated systems, the torques only for the actuated joints can be computed given the desired end-effector force, $F$, by using Eq. (20). To exclude the unactuated joints, the selection matrix, $S^k$, for $k$ actuated joints can be defined as the following if the first $(n-k)$ joints are not actuated:

$$\begin{aligned} \Gamma_{actuated} &= S^k\Gamma, \\ S^k &= [0_{k\times(n-k)} \quad I_{k\times k}], \end{aligned} \tag{21}$$

where $n$ is the number of total joints and

$$\Gamma = (S^k)^{\mathrm{T}}\Gamma_{actuated}. \tag{22}$$

Then, Eq. (20) can be written as

$$F = \bar{J}^{\mathrm{T}}(S^k)^{\mathrm{T}}\Gamma_{actuated}. \tag{23}$$

This relationship can be inverted to compute a mapping from $F$ to $\Gamma_{actuated}$ when the rank of $\bar{J}^{\mathrm{T}}(S^k)^{\mathrm{T}}$ is greater than or equal to the dimension of $F$:[7]

$$\Gamma_{actuated} = (J^k)^{\mathrm{T}}F + (N^k)^{\mathrm{T}}\Gamma_0^k. \tag{24}$$

where

$$\begin{aligned} (J^k)^{\mathrm{T}} &= \overline{(\bar{J})^{\mathrm{T}}(S^k)^{\mathrm{T}}}, \\ (N^k)^{\mathrm{T}} &= I - (J^k)^{\mathrm{T}}(\bar{J}^k)^{\mathrm{T}}. \end{aligned} \tag{25}$$

In the case of a fixed base system the equation simplifies because $S^k$ is the identity, resulting in

$$\Gamma = J^{\mathrm{T}}F + N^{\mathrm{T}}\Gamma_0. \tag{26}$$

### *5.1. Application to previous example*

In Section 4, the Jacobian and joint space inertia matrices are

$$\begin{aligned} J &= [1 \quad 1], \\ A &= \begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}, \end{aligned} \tag{27}$$

and

$$\bar{J} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad S^k = [0 \quad 1], \tag{28}$$

$$\bar{J}^{\mathrm{T}}(S^k)^{\mathrm{T}} = 1. \tag{29}$$

Now $\Gamma_{actuated}$ can be computed using Eq. (24). From Eqs. (25) and (29), $(J^k)^{\mathrm{T}} = 1$. The null-space projection matrix $N^k = 0$ because there is no redundancy in this example. Therefore,

$$\Gamma_{actuated} = F_{desired}. \tag{30}$$

## 6. Analysis of Jacobian Transpose Control Using Incorrect Mapping (Generalized Jacobian)

Having derived the precise force–torque relationship for floating base manipulators, it is worth looking back at the effects of mistakenly using the generalized Jacobian. In fact, it turns out that the errors induced by the generalized Jacobian transpose, while noticeable, are not dramatic.

Consider using Eq. (8) and analyzing its effect on Eq. (23):

$$\begin{aligned} F &= \bar{J}^{\mathrm{T}}(S^k)^{\mathrm{T}}\Gamma_{actuated} \\ &= \bar{J}^{\mathrm{T}}(S^k)^{\mathrm{T}}(J^*)^{\mathrm{T}}F_{desired}. \end{aligned} \tag{31}$$

The generalized Jacobian defined in Eq. (5) can be related with a null space projection matrix as (see the Appendix for detailed derivation)

$$(S^k)^{\mathrm{T}}(J^*)^{\mathrm{T}} = N_m^{\mathrm{T}}J^{\mathrm{T}}, \tag{32}$$

where $N_m^{\mathrm{T}}$ is the null space projection matrix associated with the linear and angular momentum. Then,

$$\begin{aligned} F &= \bar{J}^{\mathrm{T}}N_m^{\mathrm{T}}J^{\mathrm{T}}F_{desired} \\ &= (JA^{-1}J^{\mathrm{T}})^{-1}\big(JA^{-1}N_m^{\mathrm{T}}J^{\mathrm{T}}\big)F_{desired}. \end{aligned} \tag{33}$$

Using the definition of $N_m^{\mathrm{T}}$, it can be shown that the following matrix is positive semi-definite:

$$JA^{-1}J^{\mathrm{T}} - JA^{-1}N_m^{\mathrm{T}}J^{\mathrm{T}} \succeq 0. \tag{34}$$

Then,

$$I - (JA^{-1}J^{\mathrm{T}})^{-1}\big(JA^{-1}N_m^{\mathrm{T}}J^{\mathrm{T}}\big) \succeq 0. \tag{35}$$

Therefore, when the generalized Jacobian is used for the Jacobian transpose control, the magnitude of the output force on the end-effector, $||F||$, is always smaller than or equal to that of the desired force, $||F_{desired}||$:

$$||F|| \leq ||F_{desired}||, \tag{36}$$

and the angle between $F$ and $F_{desired}$ is less than or equal to $90°$:

$$F \cdot F_{desired} \geq 0. \tag{37}$$

In the end, the effect of using the generalized Jacobian is akin to having saturation, where the system could lose stability (less damping) and lose DC gain. In practice, though, systems will typically show similar performance to control using the correct mapping.

## 7. Conclusion

The torque–force mapping for space robots, or more generally for underactuated systems, is explained in this paper. The generalized Jacobian can be used for Jacobian transpose control but its transpose *does not* accurately represent the correct mapping from the end-effector (task space) force to joint space torque. It can be easily misunderstood due to the similarity with the case of fixed manipulators. Although the effect of using this incorrect mapping may not be critical to the performance or stability of the robot control, it is important to understand the property when it is applied. The correct mapping for control could be derived by inverting the mapping from joint torque to task space force.

## Appendix. Generalized Jacobian with a null space projection matrix

The goal of this appendix is to relate the generalized Jacobian, $J^*$, with the full Jacobian, $J$, and null space projection matrix, $N_m^{\mathrm{T}}$, associated with the linear and angular momentum.

The joint space inertia matrix, $A$, can be partitioned corresponding to the base and actuated joints:

$$A = \begin{bmatrix} A_{bb} & A_{ba} \\ A_{ab} & A_{aa} \end{bmatrix}, \tag{A.1}$$

where $b$ and $a$ denote *base* and *actuated*. The matrices, $A_{bb}$, $A_{ba}$, $A_{ab}$, and $A_{aa}$ are $6 \times 6$, $6 \times k$, $k \times 6$, and $k \times k$ sub-matrices where $k$ is the number of the actuated joints. The matrix, $A$, is symmetric because it is the joint space inertia matrix. The Jacobian and selection matrices are also partitioned:

$$\begin{aligned} J &= [J_b \quad J_a], \\ S^k &= [0_{k\times 6} \quad I_{k\times k}]. \end{aligned} \tag{A.2}$$

The linear and angular momentum can be expressed as[4,8]

$$\begin{aligned} m &= A_{bb}\dot{q}_{base} + A_{ba}\dot{q}_{actuated} \\ &= J_m \dot{q}, \end{aligned} \tag{A.3}$$

where

$$J_m = [A_{bb} \quad A_{ba}]. \tag{A.4}$$

Then, the corresponding null space projection matrix is

$$\begin{aligned} N_m^{\mathrm{T}} &= I - J_m^{\mathrm{T}} \bar{J}_m^{\mathrm{T}} \\ &= I - J_m^{\mathrm{T}} \left(J_m A^{-1} J_m^{\mathrm{T}}\right)^{-1} J_m A^{-1}. \end{aligned} \tag{A.5}$$

Now, the matrix, $N_m^{\mathrm{T}}$, will be evaluated so that $JN_m$ can be expressed in terms of $J^*$ and $S^k$. First, the term $(J_m A^{-1} J_m^{\mathrm{T}})$ in Eq. (A.5) can be simplified using Eq. (A.4) and the block matrix inversion of the matrix, $A$:

$$J_m A^{-1} J_m^{\mathrm{T}} = A_{bb}. \tag{A.6}$$

Therefore, the null space projection matrix, $N_m^{\mathrm{T}}$, is

$$N_m^{\mathrm{T}} = \begin{bmatrix} 0_{6\times 6} & 0_{6\times n} \\ -A_{ba}^{\mathrm{T}} A_{bb}^{-1} & I_{n\times n} \end{bmatrix}. \tag{A.7}$$

The generalized Jacobian now can be expressed as

$$\begin{aligned} JN_m &= [0 \quad -J_b A_{bb}^{-1} A_{ba} + J_a] \\ &= J^* S^k, \end{aligned} \tag{A.8}$$

or

$$(S^k)^{\mathrm{T}} (J^*)^{\mathrm{T}} = N_m^{\mathrm{T}} J^{\mathrm{T}}. \tag{A.9}$$

In Eq. (A.8), the expression of $J^* = -J_b A_{bb}^{-1} A_{ba} + J_a$ is used with zero linear and angular momentum condition.[4]

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