Dynamic Robotic Systems Lab

Generalized Inverse of a Matrix And Its Applications.

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Definition

Type(1)-Minimum norm

Type(2)-Least Squared

Type(3)-Minimum norm least squares Solution

Inverse Matrix

A is m by m nonsingular matrix, Then there exists an inverse with the property

$$AA^{-1} = A^{-1}A = I$$

Left(or Right) Inverse Matrix

A is m by n rectangular matrix, Then

Case i) rank $n \le m$, there exists a left inverse

$$A_L^{-1} = (A^T A)^{-1} A^T$$

Case ii) rank $m \le n$, there exists a right inverse

$$A_R^{-1} = A^T (AA^T)^{-1}$$

When such inverses do not exist,
can we represent a solution of the consistent equation $Ax = y$ in the form $x = Gy$?
If such a G exists, we call it a generalized inverse.

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Generalized Inverse

Three equivalent definitions of a G inverse

Definition 1.

An n by m matrix G is said to be a g inverse of an m by n matrix A if $x = Gy$ is a solution to the equation $Ax = y$ For any y such that the equation $Ax = y$ is consistent.

Definition 2.

G is a g inverse of A if $AGA = A$

Definition 3.

G is a g inverse of A if AG is idempotent* and $R(AG) =$

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Generalized Inverse

A matrix G satisfying any one of these definitions is called a g inverse.

Theorem 1.

Let $Ax = y$ be a consistent equation and A^{-} be a g inverse of A.

i) Then $x = A^{-}y$ is a solution.

ii) The class of all solutions is provided by

$x = A^{-}y + (I - A^{-}A)z$, z arbitrary

Definition

Type(1)-Minimum norm

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The g inverse for minimum norm solution

From Theorem 1-ii, the general solution is $x = Gy + (I - GA)z$.

So, There exists a choice of G independently of y such that the solution Gy has a minimum norm.

Proof. If such a G exists,

$\|Gy\| \leq \|Gy + (I - GA)z\|$, for all z and y

That is,

$$\|\mathrm{GAx}\| \leq \|\mathrm{GAx} + (\mathrm{I} - \mathrm{GA})\mathrm{z}\|$$

This implies $(\mathrm{GAx}, (\mathrm{I} - \mathrm{GA})\mathrm{z}) = 0$ for all z and x,

=> $(GA)^T(I - GA) = 0$ or $(GA)^T = (GA)$

Definition

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The g inverse for minimum norm solution

Theorem 2.

Let $Ax = y$ be a consistent equation and G be a g inverse of A such that Gy is a minimum norm solution.

i) Then, $AGA = A$, $(GA)^T = (GA)$

ii) $AGA = A$, $(GA)^{\mathrm{T}}N = N(GA)$ if $(y, x)_n = x^{\mathrm{T}}Ny$

->Minimum N norm g inverse.

Especially ***Theorem 2-ii***,
$G = N^{-1}A^T(AN^{-1}A^T)^{-1}$ satisfies the conditions of the theorem for any choice of the g inverse.

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Definition

Type(1)-Minimum norm

Type(2)-Least Squared

Type(3)-Minimum norm least squares Solution

The g inverse for a least square solution

From Theorem 1-ii, the general solution is $x = Gy + (I - GA)z$.

So, There exists a least squares solution by minimizing $\|Ax - y\|$.

Proof. If such a G exists,
$\|AGy - y\| \leq \|Ax - y\|$ for all x, y

This implies $(Aw, (AG - I)y) = 0$ for all y and x, and $w = x - Gy$ implies $A^T(AG - I) = 0$.

=> $AG = (AG)^T, AGA = A$

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Type(1)-Minimum norm

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The g inverse for minimum norm solution

Theorem 3.

Let $Ax = y$ be a consistent equation and G be a g inverse of A such that Gy is a least squares solution

i) Then, $AGA = A$, $(AG)^T = (AG)$

ii) $AGA = A$, $(AG)^T \mathrm{M} = \mathrm{M}(GA)$ if $(y, x)_m = x^{\mathrm{T}} My$

->M least squares g inverse.

Especially ***Theorem 3-ii***,

$G = (A^T MA)^{-1} A^T M$ satisfies the conditions of the theorem for any choice of the g inverse.

Definition

Type(1)-Minimum norm

Type(2)-Least Squared

Type(3)-Minimum norm least squares Solution

The g inverse for minimum norm least square solution

From Theorem 1-ii, the general solution is $x = Gy + (I - GA)z$.

So, There exists matrix G such that Gy has minimum norm in the class of least squares solutions.

Proof. If such a G exists,

$\|Gy\|_n \le \|\zeta\|_n, \{\zeta: \|A\zeta - y\|_m \le \|Ax - y\|_m\}$ for all x, y

=>$\|Gy\| \le \|\zeta\|, \{\zeta: A^T A\zeta \le A^T y\}$ for all y

This implies $A^T(I - AG) = 0$ and $G^T(I - GA) = 0$

=>$AGA = A, \left(AG\right) = (AG)^T, GAG = G, \left(GA\right) = (GA)^T$

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The g inverse for minimum norm solution

Theorem 4.

Let $Ax = y$ be a consistent equation and G be a g inverse of A such that Gy is a minimum norm solution.

i) Then, $AGA = A$, $(AG) = (AG)^T$, $GAG = G$, $(GA) = (GA)^T$

ii) $AGA = A$, $GAG = G$, $(AG)^T M = MAG, (GA)^T N = NG)$

if $(y, x)_m = x^T My$, $(y, x)_n = x^T Ny$

->Minimum N norm M least squares g inverse.

Especially ***Theorem 4-i***,

$G = A^T A(A^T AA^T A)^- A^T$ satisfies the conditions of the theorem for any choice of the g inverse.